INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids **30**: 541–555 (1999)

A PRECONDITIONING TECHNIQUE FOR STEADY EULER SOLUTIONS

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SUMMARY

This paper presents a preconditioning technique for solving a two-dimensional system of hyperbolic equations. The main attractive feature of this approach is that, unlike a technique based on simply extending the solver for a one-dimensional hyperbolic system, convergence and stability analysis can be investigated. This method represents a genuine numerical algorithm for multi-dimensional hyperbolic systems. In order to demonstrate the effectiveness of this approach, applications to solving a two-dimensional system of Euler equations in supersonic flows are reported. It is shown that the Lax–Friedrichs scheme diverges when applied to the original Euler equations. However, convergence is achieved when the same numerical scheme is employed using the same CFL number to solve the equivalent preconditioned Euler system. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: Euler equations; Lax–Friedrichs scheme; multi-dimensional hyperbolic system

1. INTRODUCTION

Numerical solutions for multi-dimensional non-linear systems of hyperbolic equations are frequently required when dealing with computational fluid dynamics problems. Although many numerical algorithms have been developed and applied to solve multi-dimensional hyperbolic systems, the majority of these methods are essentially based on simply extending the algorithms for one-dimensional problems. Strictly speaking, rigorous convergence and stability analysis are not available when applying one-dimensional numerical scheme to multi-dimensional problems. Consequently, an explicit numerical scheme is seldom employed because of the lack of stability analysis.

This paper begins by giving a brief description on numerical algorithms for a one-dimensional system of hyperbolic equations in Section 2.1. The difficulty in extending the procedure based on a one-dimensional system to multi-dimensional problems is discussed in Section 2.2. In Section 2.3, after introducing the concept of a weakly coupled system, the authors show that if a two-dimensional system of hyperbolic equations is weakly coupled, then there exists a preconditioning operator so that the resulting preconditioned matrix coefficients are hyperbolic and commutative. As a direct consequence of this property, convergence and stability analysis can be easily investigated for the equivalent preconditioned system of hyperbolic equations. The preconditioning technique is then applied for the solutions of two-dimensional

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Contract/grant sponsor: Natural Sciences and Engineering Research Council of Canada

CCC 0271–2091/99/130541–15$17.50

Received December 1997
Revised March 1998

Euler equations in Section 3. The authors first prove that the two-dimensional Euler equations are a weakly coupled hyperbolic system if and only if the flow is supersonic. Then the derivation of a preconditioning operator is presented. The preconditioning technique can be applied to Euler equations in both conservative or primitive variables. To demonstrate the effectiveness of the preconditioning technique, numerical simulations for solving two-dimensional Euler equations resulting from a supersonic channel flow problem and a shock reflection problem are reported in Section 4. Finally, concluding remarks are presented in Section 5.

2. NUMERICAL SOLUTIONS FOR HYPERBOLIC SYSTEMS

2.1. One-dimensional system

A one-dimensional linear hyperbolic system can be written as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{1}$$

where $U = (u_1, u_2, \ldots, u_n)^T$ and A is an $n \times n$ constant matrix. Since system (1) is hyperbolic, the matrix A is diagonalizable. Consequently, there exists a non-singular matrix T, such that

$$T^{-1}AT = \Lambda = \text{diag}\{\lambda_i\}, \tag{2}$$

where λ_i are the eigenvalues of A. Here, T and T^{-1} are the right and left eigenvectors of A. Now, introducing the characteristic variables $W = T^{-1}U$, the system given in (1) is equivalent to

$$\frac{\partial W}{\partial t} + \Lambda\frac{\partial W}{\partial x} = 0. \tag{3}$$

Hence, the solution of a one-dimensional hyperbolic system (1) can be obtained by solving the corresponding uncoupled system (3), which represents n simple one-way wave equations. Numerical methods for scalar one-dimensional wave equations are well-developed, the analysis of convergence and stability are also available [1–3]. This result can be extended to quasi-linear systems and non-linear conservation laws by freezing the coefficients locally at each time step.

2.2. Two-dimensional system

For a two-dimensional system, we have

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0. \tag{4}$$

The system is hyperbolic, if the eigenvalues of $\alpha A + (1-\alpha)$B are real for all α and the matrix $\alpha A + (1-\alpha)B$ is diagonalizable by a non-singular matrix S.

The main difficulty in dealing with a two-dimensional hyperbolic system is that the coefficient matrices A and B usually do not commute. Consequently, A and B can not be simultaneously diagonalizable. In general, it is not possible to rewrite the two-dimensional system (4) into a simple uncoupled form as in the case for a one-dimensional hyperbolic system.

An upwind finite difference approximation is frequently used for solving hyperbolic system of equations. In order to derive a correct upwinding direction, information on the eigenvalues of A and B is needed. For numerical computations, these eigenvalues are simply calculated by $S_1^{-1}AS_1 = \Lambda_A = \operatorname{diag}\{\lambda_i(A)\}$ and $S_2^{-1}BS_2 = \Lambda_B = \operatorname{diag}\{\lambda_i(B)\}$, where $S_1 \neq S_2$. Hence, numerical algorithms based on simply extending the results for a one-dimensional system are frequently used without the support of convergence and stability analysis.

Another technique often used for multi-dimensional hyperbolic systems is based on splitting algorithms or locally one-dimensional methods, in which the solution of system (4) is obtained by solving

$$\frac{\partial U^*}{\partial t} + A\frac{\partial U^*}{\partial x} = 0$$

and

$$\frac{\partial U^{**}}{\partial t} + B\frac{\partial U^{**}}{\partial y} = 0.$$

However, unlike applications to a two-dimensional scalar equation, where there is no splitting error, this approach introduces an error that depends on the commutator $AB - BA$. Since the matrices A and B in general do not commute, the overall accuracy of a splitting scheme is only first-order, even though high-order methods may be used to solve the corresponding one-dimensional systems. In addition, for solutions with shock waves, the resolution of the resulting numerical approximations will be less accurate since two-dimensional effects do play a strong role in the behaviour of the solutions.

Hence, although it is relatively easy to extend one-dimensional methods for two-dimensional applications, there are certainly various disadvantages associated with this approach. Therefore, it is of strong interest to develop a genuine numerical algorithm for multi-dimensional hyperbolic systems.

2.3. Preconditioning technique

In order to present a genuine algorithm for multi-dimensional hyperbolic systems, the concept of a weakly coupled system is first introduced.

Definition 1
Let A and B be $n \times n$ matrices, then A and B are said to be weakly coupled if there exists an $n \times n$ matrix K, such that

1. K is positive definite;
2. KA and KB are commutative, i.e. $KAKB = KBKA$ or $AKB = BKA$.

Consider the case $n = 2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. Without loss of generality, it is assumed that $b_2 \neq 0$. Now, introducing a matrix $K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, then by the definition of a weakly coupled system, if A and B are weakly coupled, the matrix K can be determined by solving $AKB = BKA$. Thus,

$$K = \begin{bmatrix} -k_4 - \dfrac{b_1 + b_4}{b_2} k_2 & k_2 \\ \dfrac{b_3}{b_2} k_2 & k_4 \end{bmatrix}.$$

The matrix K can be made to be positive definite by setting appropriate values for k_2 and k_4. It is important to note that even though A and B do not commute, the resulted matrices KA and KB are commutative. The concept of a weakly coupled system can easily be extended to a general problem in which the matrix A is given by

$$A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Definition 2

The hyperbolic system given in (4) is said to be a weakly coupled system if it is hyperbolic and there exists a matrix K such that

1. K is positive definite;
2. The matrices KA and KB are hyperbolic and commutative, i.e. $KAKB = KBKA$.

When dealing with a quasi-linear system, the system is defined as weakly coupled if it is weakly coupled for every locally frozen Jacobian matrix.

Now, if the system (4) is weakly coupled, it can be rewritten as

$$K\frac{\partial U}{\partial t}+KA\frac{\partial U}{\partial x}+KB\frac{\partial U}{\partial y}=0. \tag{5}$$

Suppose we are only interested in the steady solution of (4), the solution can then be computed from

$$\frac{\partial U}{\partial t}+KA\frac{\partial U}{\partial x}+KB\frac{\partial U}{\partial y}=0. \tag{6}$$

The above equations can be regarded as an equivalent preconditioned system of Equation (4), where K plays a role as a preconditioning operator. An important consequence of this is that, even though the coefficient matrices A and B are not commutative, KA and KB do commute. Since the coefficient matrices in the preconditioned system are simultaneously diagonalizable, we have

$$T^{-1}KAT=\Lambda_{KA}=\mathrm{diag}\{\lambda_i(KA)\}$$

and

$$T^{-1}KBT=\Lambda_{KB}=\mathrm{diag}\{\lambda_i(KB)\},$$

where Λ_{KA} and Λ_{KB} are diagonal matrices containing the eigenvalues of KA and KB respectively. By introducing the characteristic variables, system (6) can be rewritten in an uncoupled system as

$$\frac{\partial W}{\partial t}+\Lambda_{KA}\frac{\partial W}{\partial x}+\Lambda_{KB}\frac{\partial W}{\partial y}=0, \qquad W(x, y, 0)=W_0(x, y). \tag{7}$$

Not only that a simple and efficient numerical algorithm can be developed for the uncoupled system (7), but more important, convergence and stability analysis can also be easily established. For example, in an upwind finite difference scheme and using a semi-discretization formulation, system (7) can be expressed as

$$\frac{\mathrm{d}W}{\mathrm{d}t}=HW+b. \tag{8}$$

It is then easy to show that H is non-positive, i.e. $(HU, U) \le 0$. If $b = 0$, due to the monotonicity property, it can be proven that the solution of (8) satisfies the following relation

$$(W(t), W(t)) \le (W(s), W(s)), \quad \text{for } t > s \ge 0.$$

Hence, $\|U(t)\| \le C\|U(0)\|$, where $U(t)$ is a numerical approximation to system (4). If $b \ne 0$, the general solution of (8) has the form

$$W(t) = S(t)W_0 + \int_0^t S(t-s)\,\mathrm{d}s,$$

where $S(t)$ is the semi-group e^{Ht}.

Using the classical von Neumann stability analysis, it can be shown that an explicit upwind difference scheme is stable if and only if

$$(\sigma_x \rho(KA) + \sigma_y \rho(KB)) \le 1,$$

where $\sigma_x = \Delta t/\Delta x$, $\sigma_y = \Delta t/\Delta y$, $\rho(KA)$ and $\rho(KB)$ are the spectral radius of KA and KB matrices. For the explicit Lax–Friedrichs scheme, it is stable if and only if

$$(\sigma_x \rho(KA))^2 + (\sigma_y \rho(KB))^2 \le \frac{1}{2}.$$

3. APPLICATIONS TO EULER EQUATIONS

In a primitive variable formulation, the two-dimensional Euler equations can be expressed by system (4), where

$$U = \begin{pmatrix} \rho \\ u \\ v \\ P \end{pmatrix}, \qquad A = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \rho^{-1} \\ 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & u \end{pmatrix}, \qquad B = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \rho^{-1} \\ 0 & 0 & \rho c^2 & v \end{pmatrix}.$$

Here, ρ denotes the density, u and v are the velocities in the x- and y-directions respectively, P is the pressure and c is the speed of sound.

Theorem
The two-dimensional Euler equation is a weakly coupled hyperbolic system if and only if the flow is supersonic, i.e. $(u^2 + v^2)/c^2 > 1$.

Proof
The proof of this theorem will be completed by using the following lemmas.

Lemma 1
Let K be a 4×4 matrix with coefficients k_{ij}, then by solving the matrix equation $AKB = BKA$, the general solution is given by

$$K = \begin{pmatrix} k_{11} & t_1 & \dfrac{v}{u} t_1 - \dfrac{\rho^2 v}{u} t_3 + \rho^2 t_6 & t_2 \\ \dfrac{u}{v} t_4 & k_{22} & k_{23} & t_3 \\ t_4 & k_{23} & t_5 & t_6 \\ 0 & \rho^2 c^2 t_3 + \dfrac{\rho^2 u}{v} t_4 & \rho^2 t_4 + \rho^2 c^2 t_6 & k_{44} \end{pmatrix},$$

where

$$k_{11} = -c^2 t_2 - \rho u t_3 - \frac{\rho(u^2+v^2)}{c^2 v} t_4 - \rho v t_6,$$

$$k_{22} = -\frac{\rho u(c^2 - u^2 + v^2)}{v^2} t_3 + \frac{\rho(u^2 - v^2)(c^2 - u^2 - v^2)}{c^2 v^3} t_4 - \frac{u^2}{v^2} t_5,$$

$$k_{23} = \frac{\rho(u^2 - c^2)}{v} t_3 + \frac{\rho u(u^2 + v^2 - c^2)}{c^2 v^2} t_4 + \frac{u}{v} t_5 + \rho u t_6,$$

$$k_{44} = -\rho u t_3 - \frac{\rho(u^2+v^2)}{c^2 v} t_4 - \rho v t_6,$$

in which $t_1, t_2, \ldots, t_6$ are parameters.

Lemma 2
The first two eigenvalues of KA and KB are given by

$$\lambda_{1,2}(KA) = \rho(c^2 - u^2)t_3 - \frac{\rho u(u^2 + v^2 - c^2)t_4}{vc^2} - \rho uvt_6 \pm \frac{\rho\sqrt{u^2 + v^2 - c^2}}{c}(t_4 + c^2 t_6),$$

and

$$\lambda_{1,2}(KB) = -\rho uvt_3 + \rho c^{-2}(c^2 - u^2 - v^2)t_4 + \rho(c^2 - v^2)t_6 \pm \frac{\rho\sqrt{u^2 + v^2 - c^2}}{vc}(vc^2 t_3 + ut_4).$$

Remark
From lemma 2, the necessary condition for the theorem is established. Notice that if KA and KB are hyperbolic, then both $\lambda_1(KA)$ and $\lambda_1(KB)$ must be real. Therefore, a necessary condition is that $u^2 + v^2 - c^2 > 0$ or $(u^2 + v^2)/c^2 > 1$. Conversely, if $u^2 + v^2 - c^2 < 0$, the following two conditions,

$$t_4 + c^2 t_6 = 0$$

and

$$vc^2 t_3 + ut_4 = 0,$$

must be satisfied. However, it is not hard to verify that these conditions will make K become singular, since all coefficients in the last row of K are zero. Hence, it is concluded that $u^2 + v^2 - c^2 > 0$.

Lemma 3
Now, setting the parameters $t_1, t_2, \ldots, t_6$ as

$$\begin{aligned}
t_1 &= 0,\\
t_2 &= 0,\\
t_3 &= -\frac{u}{\rho q^2},\\
t_4 &= 0,\\
t_5 &= \frac{q^2 - c^2 + sv^2}{q^2},\\
t_6 &= -\frac{v}{\rho q^2},
\end{aligned}$$

where s is a parameter, the matrix K in lemma 1 can be rewritten as

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \dfrac{c^2}{q^2} + s\dfrac{u^2}{q^2} & s\dfrac{uv}{q^2} & -\dfrac{u}{\rho q^2} \\ 0 & s\dfrac{uv}{q^2} & 1 - \dfrac{c^2}{q^2} + s\dfrac{v^2}{q^2} & -\dfrac{v}{q^2} \\ 0 & -\rho c^2\dfrac{u}{q^2} & -\rho c^2\dfrac{v}{q^2} & 1 \end{pmatrix}.$$

Here $q^2 = v^2 + u^2$. Now, if $(u^2 + v^2)/c^2 = q^2/c^2 > 1$, then K is a positive definite matrix by setting an appropriate value for s.

Proof
In order to show K is positive definite, we need to show the symmetric part matrix $(K + K^T)/2$ is positive definite. Now,

$$\frac{K + K^T}{2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \dfrac{c^2}{q^2} + s\dfrac{u^2}{q^2} & s\dfrac{uv}{q^2} & -\dfrac{1 + \rho^2 c^2}{2\rho q^2}u \\ 0 & s\dfrac{uv}{q^2} & 1 - \dfrac{c^2}{q^2} + s\dfrac{v^2}{q^2} & -\dfrac{1 + \rho^2 c^2}{2\rho q^2}v \\ 0 & -\dfrac{1 + \rho^2 c^2}{2\rho q^2}u & -\dfrac{1 + \rho^2 c^2}{2\rho q^2}v & 1 \end{pmatrix}.$$

Clearly, it is positive definite provided

(i) $D_1 = 1 > 0$,

(ii) $D_2 = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 - \dfrac{c^2}{q^2} + s\dfrac{u^2}{q^2} \end{pmatrix} > 0,$

(iii) $D_3 = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \dfrac{c^2}{q^2} + s\dfrac{u^2}{q^2} & s\dfrac{uv}{q^2} \\ 0 & s\dfrac{uv}{q^2} & 1 - \dfrac{c^2}{q^2} + s\dfrac{v^2}{q^2} \end{pmatrix} > 0,$

(iv) $D_4 = \det\left(\dfrac{K + K^T}{2}\right) > 0.$

Notice that D_1 is naturally satisfied. Since

$$D_2 = 1 - \frac{c^2 - su^2}{q^2},$$

it is easy to see that $D_2 > 0$ by setting $s > 0$. Now

$$D_3 = \left(1 - \frac{c^2}{q^2}\right)^2 + s\left(1 - \frac{c^2}{q^2}\right),$$

and $D_3 > 0$ if we let $s > 0$. Finally, D_4 can be expressed as

$$D_4 = \left(1 - \frac{c^2}{q^2}\right)\left(s - \frac{(1 + \rho^2 c^2)^2}{4\rho^2 q^2} + 1 - \frac{c^2}{q^2}\right).$$

If $s \geq [(1 + \rho^2 c^2)^2/4\rho^2 q^2] - 1 + (c^2/q^2)$, then $D_4 > 0$. Since $q^2/c^2 > 1$, one can choose $s = [(1 + \rho^2 c^2)^2/4\rho^2 q^2]$.

Lemma 4
The matrix K defined in the previous lemma commutes with both A and B.

Proof
Since K is positive definite, it is not hard to verify that $AKB = BKA$. Alternatively, let the matrices be rewritten as $A = uI + A_1$ and $B = vI + B_1$, where I is an identity matrix and

$$A_1 = \begin{pmatrix} 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 1/\rho \\ 0 & 0 & 0 & 0 \\ 0 & \rho c^2 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho \\ 0 & 0 & \rho c^2 & 0 \end{pmatrix}.$$

Then, it is easy to verify that

$$uKB_1 + vA_1K + A_1KB_1 = vKA_1 + uB_1K + B_1KA_1.$$

Notice that the above condition is equivalent to showing $AKB = BKA$.

Lemma 5
Let K be defined as in lemma 3, then the eigensystems for KA and KB are as follows.

1. The eigenvalue matrix of KA is

$$\Lambda_{KA} = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u\left(1 - \dfrac{2c^2}{q^2} + s\right) & 0 & 0 \\ 0 & 0 & u\left(1 - \dfrac{c^2}{q^2}\right) - \dfrac{c^2 v\alpha}{q^2} & 0 \\ 0 & 0 & 0 & u\left(1 - \dfrac{c^2}{q^2}\right) + \dfrac{c^2 v\alpha}{q^2} \end{bmatrix},$$

2. The eigenvalue matrix of KB is

$$\Lambda_{KB} = \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & v\left(1 - \dfrac{2c^2}{q^2} + s\right) & 0 & 0 \\ 0 & 0 & v\left(1 - \dfrac{c^2}{q^2}\right) + \dfrac{c^2 u\alpha}{q^2} & 0 \\ 0 & 0 & 0 & v\left(1 - \dfrac{c^2}{q^2}\right) - \dfrac{c^2 u\alpha}{q^2} \end{bmatrix},$$

3. Their eigenvectors are, in either case

$$L = \begin{bmatrix} 1 & \dfrac{1}{sq^2 - 2c^2} & \dfrac{1}{2c^2} & \dfrac{1}{2c^2} \\ 0 & \dfrac{u}{\rho q^2} & -\dfrac{u + v\alpha}{2\rho q^2} & \dfrac{v\alpha - u}{2\rho q^2} \\ 0 & \dfrac{v}{\rho q^2} & \dfrac{u\alpha - v}{2\rho q^2} & -\dfrac{u\alpha + v}{2\rho q^2} \\ 0 & 0 & \dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix},$$

where $\alpha = \sqrt{(q^2/c^2) - 1} > 0$.

Notice that the above five lemmas imply the theorem. Lemma 2 gives the necessary condition for the theorem, and the 'if' part of the theorem is proved using Lemmas 3–5.

The preconditioning technique can also be easily extended to two-dimensional Euler equations in conservative variables,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0,$$

where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix},\qquad F(U)=\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ (E+P)u\end{pmatrix},\qquad G(U)=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ (E+P)v\end{pmatrix}.$$

Here E denotes the energy. The Jacobian matrix of the transformation from the non-conservative variables $V=(\rho, u, v, P)^T$ to the conservative variables U is given by

$$M=\frac{\partial V}{\partial U}=\begin{pmatrix}1 & 0 & 0 & 0\\ -u\rho^{-1} & \rho^{-1} & 0 & 0\\ -v\rho^{-1} & 0 & \rho^{-1} & 0\\ \frac{(\gamma-1)}{2}(u^2+v^2) & -(\gamma-1)u & -(\gamma-1)v & \gamma-1\end{pmatrix}.$$

Now, let $A_c=M^{-1}AM$ and $B_c=M^{-1}BM$, then the quasi-linear form of a two-dimensional Euler equation in conservative variable can be written as

$$\frac{\partial U}{\partial t}+A_c\frac{\partial U}{\partial x}+B_c\frac{\partial U}{\partial y}=0, \tag{10}$$

where

$$A_c=\begin{pmatrix}0 & 1 & 0 & 0\\ \frac{\gamma-3}{2}u^2+\frac{\gamma-1}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1\\ -uv & v & u & 0\\ a_1 & a_2 & -(\gamma-1)uv & \gamma u\end{pmatrix},$$

$$B_c=\begin{pmatrix}0 & 0 & 1 & 0\\ -uv & v & u & 0\\ \frac{\gamma-3}{2}v^2+\frac{\gamma-1}{2}u^2 & -(\gamma-1)u & (3-\gamma)v & \gamma-1\\ b_1 & -(\gamma-1)uv & b_2 & \gamma v\end{pmatrix},$$

$$a_1=-\frac{\gamma uE}{\rho}+(\gamma-1)u(u^2+v^2),$$

$$a_2=\frac{\gamma E}{\rho}-\frac{\gamma-1}{2}(v^2+3u^2),$$

$$b_1 = -\frac{\gamma v E}{\rho} + (\gamma - 1)v(u^2 + v^2),$$

$$b_2 = \frac{\gamma E}{\rho} - \frac{\gamma - 1}{2}(u^2 + 3v^2).$$

It can be shown that there exists a preconditioning operator K_c, where

$$K_c = M^{-1}KM = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u\left(\dfrac{c^2}{q^2} - t_1\right) & 1 + \dfrac{t_2u^2 - c^2}{q^2} & \dfrac{t_2uv}{q^2} & \dfrac{(1-\gamma)u}{q^2} \\ v\left(\dfrac{c^2}{q^2} - t_1\right) & \dfrac{t_2uv}{q^2} & 1 + \dfrac{t_2v^2 - c^2}{q^2} & \dfrac{(1-\gamma)v}{q^2} \\ t_4 & u(t_2 - t_3) & v(t_2 - t_3) & 2 - \gamma \end{pmatrix},$$

in which $t_1 = [(\gamma - 1)/2] - s$, $t_2 = s + \gamma - 1$, $t_3 = \gamma c^2/[(\gamma - 1)q^2]$ and $t_4 = ((3 - \gamma)/2 - (1 + s - t_3)\rho)q^2$. By choosing an appropriate value for s, we can make K_c be positive definite. The steady solution of system (10) can then be obtained by solving the corresponding preconditioned Euler system

$$\frac{\partial U}{\partial t} + K_cA_c\frac{\partial U}{\partial x} + K_cB_c\frac{\partial U}{\partial y} = 0. \tag{11}$$

It is not hard to verify that the matrices K_cA_c and K_cB_c are commutative, i.e. $K_cA_cK_cB_c = K_cB_cK_cA_c$.

The details of the proof of the above theorem and the derivation of the preconditioning operator can be found in [4].

4. COMPUTATIONAL EXPERIMENTS

This section investigates the performance and effectiveness of the preconditioning technique presented in the previous sections. Intending to illustrate the main feature related to the convergence and stability analysis stated, the authors consider the steady solutions of two-dimensional Euler equations in supersonic flows,

$$\frac{\partial U}{\partial t} + \tilde{A}\frac{\partial U}{\partial x} + \tilde{B}\frac{\partial U}{\partial y} = 0, \tag{12}$$

where $\tilde{A} = KA$, $\tilde{B} = KB$ and K is the preconditioning operator. When $K = I$, the system of equations (12) reverts to the original Euler equations. The solution of Equation (12) is solved by the following numerical procedures:

Lax–Friedrichs method

Let $u^n_{i,j}$ be the approximation to $u(x_i, y_j, t^n)$, and let Δt, Δx and Δy denote the time step and mesh size in the x- and y-directions respectively. The Lax–Friedrichs scheme for the system (12) can be written as

$$U_{i,j}^{n+1} = \frac{1}{4}(U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^n) - \frac{\Delta t}{2\Delta x}\tilde{A}(U_{i+1,j}^n - U_{i-1,j}^n)$$
$$-\frac{\Delta t}{2\Delta y}\tilde{B}(U_{i,j+1}^n - U_{i,j-1}^n). \tag{13}$$

Upwinding method

Let W be the characteristic variables, then the preconditioned Euler equation (12) can be rewritten in the form

$$\frac{\partial W}{\partial t} + \Lambda_{\tilde{A}}\frac{\partial W}{\partial x} + \Lambda_{\tilde{B}}\frac{\partial W}{\partial y} = 0, \tag{14}$$

where $\Lambda_{\tilde{A}}$ and $\Lambda_{\tilde{B}}$ are diagonal matrices. Following the discussion in the previous sections, a non-singular matrix L exists and it can be used to simultaneously diagonalize both matrix coefficients $\tilde{A}$ and $\tilde{B}$. Now, define $|\tilde{A}| = L|\Lambda_{\tilde{A}}|L^{-1}$, $\tilde{A}^+ = L\Lambda_{\tilde{A}}^+ L^{-1}$, $\tilde{A}^- = L\Lambda_{\tilde{A}}^- L^{-1}$, and similar definitions are used for $\tilde{B}^+$ and $\tilde{B}^-$. Then, an upwind formula for the system (14) can be expressed in the form

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \frac{\tilde{A}^+(U_{i,j}^* - U_{i-1,j}^*) + \tilde{A}^-(U_{i+1,j}^* - U_{i,j}^*)}{\Delta x}$$
$$+ \frac{\tilde{B}^+(U_{i,j}^* - U_{i,j-1}^*) + \tilde{B}^-(U_{i,j+1}^* - U_{i,j}^*)}{\Delta y} = 0, \tag{15}$$

where $U^* = U^n$, if the upwinding scheme is explicit; and if $U^* = U^{n+1}$, if the scheme is implicit. For an implicit upwind scheme, the solution U^{n+1} is obtained by solving a sparse matrix in each time step. In this paper, the resulting sparse system is solved by the method of conjugate gradients [5].

The preconditioning technique is then applied to two test cases, one from the supersonic channel flow problem and the other from the shock reflection problem.

4.1. Results for the supersonic channel flow problem

In the first example, the authors consider the solution of supersonic flows in a channel with a 4% thick circular arc bump. The geometry of the physical space is mapped into a computational space, in which a uniform mesh 96×32 is used for numerical computations. The inflow boundary is prescribed with a supersonic Mach number $M = 1.4$ and pressure coefficient $P = 1.0$. The outflow boundary conditions are derived from the standard first-order extrapolation method. Such a problem is also used as a test case by Spekreijse [6].

The supersonic channel flow problem is then solved by the Lax–Friedrichs and upwinding methods applied to the preconditioned Euler system. In Figures 1 and 2, the Mach number distribution along the lower surface of the channel is displayed along with iso-Mach lines obtained by the Lax–Friedrichs scheme. Note that, since both upwind and Lax–Friedrichs methods are first-order-accurate, the numerical results using the upwind method are almost the same as those reported in Figures 1 and 2. The results presented here are also in good agreement with the upwinding scheme used by Spekreijse. In [6], Spekreijse solved the same problem using a first-order-accurate upwind scheme and the defect correction method for second-order accuracy. The numerical procedure employed in [6] was based on a multi-grid technique in order to achieve computational efficiency. However, it should be pointed out that

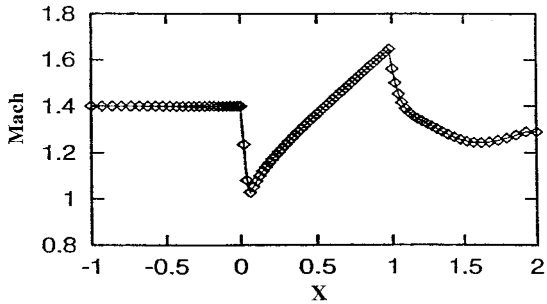

Figure 1. Mach number distributions along the lower surface.

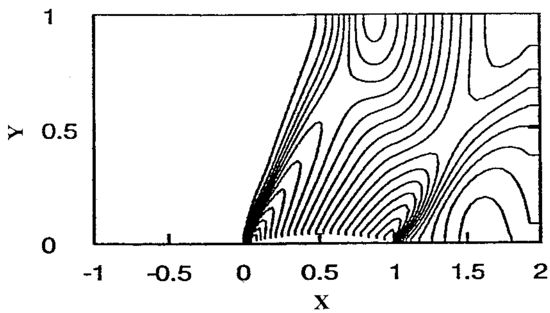

Figure 2. Mach contours.

the goal of the present paper is to present a genuine numerical algorithm for multi-dimensional hyperbolic systems, the other considerations, such as improvement on solution accuracy and computational efficiency although are important, but they are not the main objective of the present paper. The main contribution of the present paper is to introduce a preconditioning technique for a weakly coupled hyperbolic system, so that convergence and stability analysis can be easily investigated. The theoretical part has been discussed in the previous sections, and the importance in using a preconditioning operator will now be demonstrated. Let the error at the nth step be defined as the difference between the numerical solutions U^n and U^{n-1}; in Figures 3 and 4 the authors illustrate the error using the Lax–Friedrichs scheme with

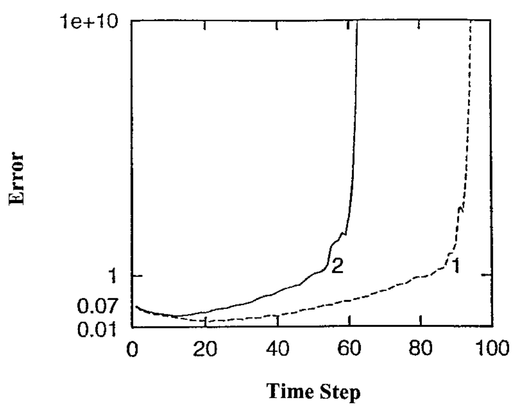

Figure 3. Numerical solution for the original Euler system; error $= \|U^n - U^{n-1}\|_\infty$; 1 represents $\Delta t/h = 0.25$; 2 represents $\Delta t/h = 0.275$.

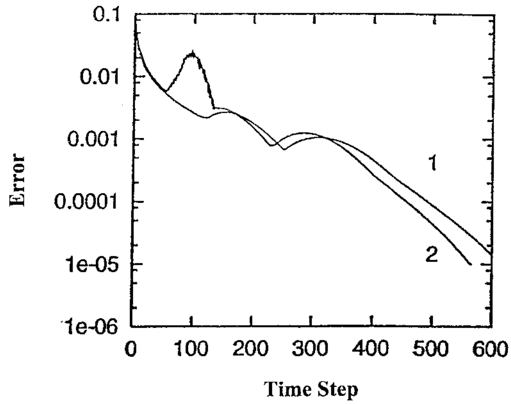

Figure 4. Numerical solution for the preconditioned Euler system; error $= \|U^n - U^{n-1}\|_\infty$; 1 represents $\Delta t/h = 0.25$; 2 represents $\Delta t/h = 0.275$.

$\Delta t/h = 0.25$ and 0.275 applied to both the original and the preconditional Euler systems (12). The advantage of the present preconditioning approach is clearly evident, in which it is observed that using the same CFL number for numerical computations, convergence is obtained only when the preconditioned Euler equations are solved. The supersonic channel flow problem has also been tested with inflow boundary conditions given at $M = 1.6$ and 2.0.

4.2. Results for the shock reflection problem

This test case provides a comparison of the solutions obtained by explicit and implicit schemes. The physical domain for the shock reflection problem is defined over a rectangle $[0, 4.1] \times [0, 1]$. The free-stream Mach number is given by $M = 2.9$, the boundary conditions at free-stream and the upper boundary are prescribed as follows:

$$\begin{pmatrix} \rho_\infty \\ u_\infty \\ v_\infty \\ P_\infty \end{pmatrix} = \begin{pmatrix} 1.0 \\ 2.9 \\ 0.0 \\ 0.714286 \end{pmatrix}; \qquad \begin{pmatrix} \rho_u \\ u_u \\ v_u \\ P_u \end{pmatrix} = \begin{pmatrix} 1.6996 \\ 2.6193 \\ -0.5062 \\ 1.5282 \end{pmatrix}.$$

The numerical solution is obtained using an upwind scheme (15) applied to a uniform grid system of 120×40. For an explicit scheme, we let $\Delta t = 0.25\Delta x$. A much larger time step, such as $\Delta t = 100\Delta x$ or $\Delta t = 500\Delta x$, can be used when the implicit version is used. In Figure 5, the pressure contour using the implicit solution with $\Delta t = 100\Delta x$ is displayed, and according to the jump conditions, the expected angle of incident 29° is clearly observed.

5. CONCLUDING REMARKS

The purpose of the present work was to draw attention on the development of a genuine numerical algorithm for multi-dimensional systems of hyperbolic equations. In this paper, the authors presented a new method with detailed theoretical analysis and derivation. The method is based on introducing a preconditioning operator to the multi-dimensional hyperbolic

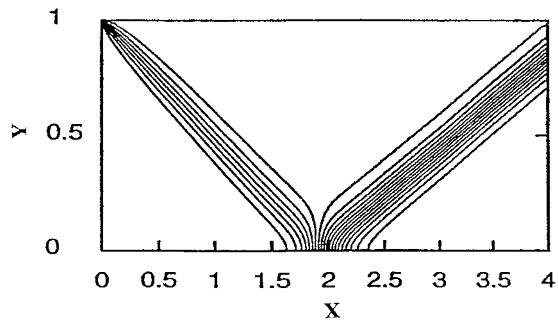

Figure 5. Pressure contours.

system. It began by defining the concept of a weakly coupled hyperbolic system, and it was shown that when a preconditioning operator is applied to such a two-dimensional weakly coupled system, there exists a non-singular matrix that can simultaneously diagonalize both matrix coefficients. As a result, a simple numerical scheme can be derived for the preconditioned hyperbolic system, in which convergence and stability analysis can easily be investigated. The new preconditioning method is then applied to solve steady Euler equations. Based upon the theoretical analysis, it was shown that the system of Euler equations in supersonic flows is a weakly coupled hyperbolic system. Consequently, the benefits with respect to convergence and stability analysis of the proposed preconditioning technique are observed for solving systems of steady Euler equations. These theoretical predictions are confirmed by computational experiments for the solutions of the supersonic channel flow problem and the shock reflection problems. The main attractive feature is clearly demonstrated from examining the convergences of the explicit Lax–Friedrichs scheme using the same CFL number applied to the original and the preconditioned systems of Euler equations. Not only a converged solution is obtained, but the stability analysis can easily be performed for the preconditioned Euler system. When an implicit upwind scheme is applied to the preconditioned system, a much larger time step can also be used without loosing the stability.

The preconditioning technique in this paper was presented only in the context of two-dimensional weakly coupled hyperbolic systems. In future studies, the authors' investigations will be extended to three-dimensional hyperbolic systems, and the results will be reported in a later paper.

ACKNOWLEDGMENTS

This research is supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The authors would like to thank the reviewers for many helpful comments on the manuscript.

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